FACTORING COMPACT OPERATORS

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ABSTRACT

We consider the following problem: Does there exist a separable Banach space Z such that every compact operator can be factored as a product TS with T, S compact, range S = Domain $T = Z$? Our investigation yields a reasonable partial solution to this problem as well as the following independent result: A Banach space which has the λ-metric approximation property can be embedded as a complemented subspace of a $\pi_{\lambda'}$ space.

1. Introduction

Let X, Y, and Z be Banach spaces and let $C(X, Y)$ be the uniform closure in the space of bounded linear operators from X to Y of the set $X^* \otimes Y$ of continuous linear operators of finite rank from X to Y. We note that $C(X, Y)$ is the space of compact operators from X to Y if either X^* or Y has the approximation property. An operator T in $C(X, Y)$ is said to factor compactly through Z (written $T \in C_Z(X, Y)$) provided there are $A \in C(X, Z)$ and $B \in C(Z, Y)$ such that $BA = T$. (A, B) is then said to be a compact factorization of T through Z.

It turns out that under certain conditions on Z, there is a natural norm on $C_Z(X, Y)$ which makes $C_Z(X, Y)$ into a Banach space (Proposition 1). The conditions on Z are fairly general; in particular, Z can be l_p, $L_p[0, 1]$, or a certain space isomorphic to $C[0, 1]$.

Proposition 1 and its variant Proposition 2 have several interesting consequences:

THEOREM 1. *There exists a family* $\{C_p: 1 \leqq p \leqq \infty\}$ *of pairwise totally incomparable separable Banach spaces each of which has the compact factorization property. (We say that* Z *has the compact factorization property if for each pair* (X, Y) *of Banach spaces,* $C(X, Y) = C_Z(X, Y)$. X *and* Y *are totally incomparable* [11] *provided no infinite dimensional subspace of* X *is isomorphic to a subspace of* Y.)

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THEOREM 2. *Let $1 \leqq p \leqq \infty$. A Banach space X is an $\mathscr{L}_p$ space (if $p = 1$ or ∞) or either an $\mathscr{L}_p$ space or an $\mathscr{L}_2$ space (if $1 < p < \infty$) if and only if for each Banach space Y, $C(Y,X) = C_{l_p}(Y, X)$. (See [3] and [4] for the definition and properties of $\mathscr{L}_p$ spaces.)*

THEOREM 3. *Suppose that either X is reflexive or that X^* is separable, let $1 \leqq p < \infty$, and let Y be an $\mathscr{L}_p$ space. Each p-absolutely summing operator from X to Y admits a compact factorization (A,B) through l_p such that A is p-absolutely summing. (See [10], [9], or [3] for the definition and properties of p-absolutely summing operators.)*

The spaces C_p mentioned in Theorem 1 also arise naturally in another context.

THEOREM 4. *Suppose that X has the λ-metric approximation property for some $\lambda \geqq 1$ and let $1 \leqq p \leqq \infty$. Then $X \oplus C_p$ is a $\pi_{\lambda'}$ space for some $\lambda' \geqq 1$. (The fact that we shall need is that X has the $\lambda + \varepsilon$ metric approximation property for every $\varepsilon > 0$ [resp. X is a $\pi_{\lambda+\varepsilon}$ space for every $\varepsilon > 0$] if and only if for each finite dimensional subspace E of X and $\varepsilon > 0$ there exists an operator [resp. projection] T of finite rank on X such that $\| T \| \leqq \lambda + \varepsilon$ and $T(x) = x$ for each $x \in E$. See [2], especially Lemma [3], [1], and references therein.)*

Note that it is a consequence of Theorem 4 that a Banach space has the λ-metric approximation property for some $\lambda \geqq 1$ if and only if it can be embedded isomorphically as a complemented subspace of a $\pi_{\lambda'}$ space for some $\lambda' \geqq 1$.)

Generally our notation follows that of [3]. We assume all Banach spaces are real, but there are no difficulties in passing to the complex case. Our main departure from standard notation lies in the use of the symbol l_∞ for the space usually denoted by c_0. "Operator" always means "bounded linear operator"; "subspace" means "closed subspace". The range (respectively, nullspace) of an operator, L, is denoted by $\mathscr{R}(L)$ (respectively, ker L). If L is an operator on X and Y is a subspace of X, the restriction of L to Y is denoted by $L_{|Y}$. "sp A" means "linear span of A"; "cl" is used for the topological closure operation. If X and Y are isomorphic Banach spaces, then $d(X, Y) = \inf\{\| T \| \, \| T^{-1} \| : T$ is an isomorphism from X onto $Y\}$. We abbreviate "if and only if" to "iff".

2. $C_Z(X, Y)$ is a Banach space

For $1 \leqq p < \infty$, let l_p be the Banach space of all real sequences $\{x_i\}_{i=1}^\infty$ such that $\| \{x_i\}_{i=1}^\infty \|_p = (\sum_{i=1}^\infty | x_i |^p)^{1/p} < \infty$. Let l_∞ be the Banach space of all real sequences $\{x_i\}_{i=1}^{\infty}$ which converge to 0 with the norm $\|\{x_i\}_{i=1}^{\infty}\|_{\infty} = \sup\{|x_i|\}_{i=1}^{\infty}$. Thus l_∞ is the Banach space which is usually denoted by c_0. We use the symbol l_∞ instead of c_0 so as to simplify the notation in several of the theorems.

Let $\{Z_i\}_{i=1}^{\infty}$ be a sequence of Banach spaces and let $1 \leqq p \leqq \infty$. We use the symbol $\Sigma_p Z_i$ or $(Z_1 \oplus Z_2 \oplus Z_3 \oplus \cdots)_p$ to denote the Banach space of all sequences $\{z_i\}_{i=1}^{\infty}$ such that $z_i \in Z_i$ and $\{\| z_i \|\}_{i=1}^{\infty} \in l_p$, where $\| \{z_i\}_{i=1}^{\infty} \| = \| \{\| z_i \|\}_{i=1}^{\infty} \|_p$. If each Z_i is isometric to Z, we sometimes write $\Sigma_p Z_i$ as $\Sigma_p Z$.

In this section we assume that $Z \cong \Sigma_p Z$, where "$X \cong Y$" means that X and Y are isometric Banach spaces. In particular, Z can be l_p $(1 \leqq p \leqq \infty)$ or $L_p[0,1]$ $(1 \leqq p < \infty)$. It is easily seen that if Z is any Banach space, then $\Sigma_p(\Sigma_p Z) \cong \Sigma_p Z$. It is known that if S is a compact metric space, then $C(S)$ is isomorphic to $\Sigma_\infty C(S)$ (see, e.g., [7]). Thus Z can be a space isomorphic to $C(S)$; namely, let Z be $\Sigma_\infty C(S)$.

In order to prove that $C_Z(X,Y)$ can be made into a Banach space when $Z \cong \Sigma_p Z$ we need one well-known lemma.

LEMMA 1. *Let k_1 and k_2 be positive numbers, let $1 < p < \infty$, and let $1/p + 1/q = 1$. Then there are positive numbers a_1, a_2, b_1 and b_2 such that $a_1 b_1 = k_1$, $a_2 b_2 = k_2$, and $(a_1^p + a_2^p)^{1/p}(b_1^q + b_2^q)^{1/q} = k_1 + k_2$.*

PROOF. Set $a_1 = k_1^{q/(p+q)}$, $a_2 = k_2^{q/(p+q)}$, $b_1 = k_1^{p/(p+q)}$, $b_2 = k_2^{p/(p+q)}$. Q.E.D.

PROPOSITION 1. *Let X, Y, and Z be Banach spaces, let $1 \leqq p \leqq \infty$, and suppose that $Z \cong \Sigma_p Z$. Then $C_Z(X,Y)$ is a Banach space under the norm $\| L \|_Z = \inf\{\| A \| \, \| B \| : (A,B)$ is a compact factorization of L through $Z\}$.*

PROOF. $C_Z(X,Y)$ is obviously closed under scalar multiplication, so to show that $C_Z(X,Y)$ is a linear space, it is sufficient to show that whenever L_1 and L_2 are in $C_Z(X,Y)$, then also $L_1 + L_2$ is in $C_Z(X,Y)$. Write $Z = (Z_1 \oplus Z_2 \oplus Z_3 \oplus \cdots)_p$, where $Z_i \cong Z$. Let P_i be the (norm 1) projection of Z onto Z_i with $\ker P_i = \operatorname{clsp} \cup_{j \neq i} Z_j$. Let (A_1, B_1) be a compact factorization of L_1 through Z_1 and let (A_2, B_2) be a compact factorization of L_2 through Z_2. Then $(A_1 + A_2, B_1 P_1 + B_2 P_2)$ is a compact factorization of $L_1 + L_2$ through Z.

We next show that $\| \cdot \|$ is a norm. The only difficulty lies in showing that $\| \cdot \|_Z$ satisfies the triangle inequality. Let L_1 and L_2 be in $C_Z(X,Y)$. We can assume that neither L_1 nor L_2 is the zero operator. Let $\varepsilon > 0$. For $i = 1,2$, let (A_i, B_i) be a compact factorization of L_i through Z_i such that $\| A_i \| \, \| B_i \| \leqq \| L_i \|_Z + \varepsilon/2$. Let $k_i = \| A_i \| \, \| B_i \|$. Note that by multiplying A_i by λ_i and B_i by $1/\lambda_i$, $\| A_i \|$ and $\| B_i \|$ are constrained only by the condition that $\| A_i \| \| B_i \| = k_i$.

Now $L_1 + L_2 = (B_1P_1 + B_2P_2)(A_1 + A_2)$, so $\| L_1 + L_2 \|_Z \leqq$

$$\| B_1P_1 + B_2P_2 \| \cdot \| A_1 + A_2 \| .$$

If $p = \infty$, $\| A_1 + A_2 \| = \max\{\| A_1 \|, \| A_2 \|\}$. Thus if we set $\| A_1 \| = \| A_2 \|$, then $\| L_1 + L_2 \|_Z \leqq (\| B_1 \| \| P_1 \| + \| B_2 \| \| P_2 \|) \| A_1 \| =$

$$\| B_1 \| \| A_1 \| + \| B_2 \| \| A_2 \|$$

$\leqq \| L_1 \|_Z + \| L_2 \|_Z + \varepsilon$.

If $p = 1$, then $\| B_1P_1 + B_2P_2 \| = \max\{\| B_1 \|, \| B_2 \|\}$. Thus if we set $\| B_1 \| = \| B_2 \|$ then

$\| L_1 + L_2 \|_Z \leqq \| B_1 \| (\| A_1 \| + \| A_2 \|) = \| B_1 \| \| A_1 \| + \| B_2 \| \| A_2 \| \leqq \| L_1 \|_Z + \| L_2 \|_Z + \varepsilon$.

We now turn to the case where $1 < p < \infty$. For $i = 1,2$, let $b_i = \| B_i \|$ and let $a_i = \| A_i \|$. Then $\| A_1 + A_2 \| \leqq (a_1{}^p + a_2{}^p)^{1/p}$ and $\| B_1P_1 + B_2P_2 \| \leqq \sup\{b_2\alpha + b_1(1 - \alpha^p)^{1/p} : 0 \leqq \alpha \leqq 1\} = (b_2^q + b_1^q)^{1/q}$, where $1/p + 1/q = 1$. Thus $\| L_1 + L_2 \|_Z \leqq (a_1{}^p + a_2{}^p)^{1/p}(b_1{}^q + b_2{}^q)^{1/q}$. By Lemma 1, we can choose a_i and b_i so that $(a_1{}^p + a_2{}^p)^{1/p}(b_1{}^q + b_2{}^q)^{1/q} = k_1 + k_2$. Thus

$$\| L_1 + L_2 \|_Z \leqq k_1 + k_2 \leqq \| L_1 \|_Z + \| L_2 \|_Z + \varepsilon .$$

This completes the proof that $\| \cdot \|_Z$ is a norm on $C_Z(X, Y)$.

Finally we show that $C_Z(X, Y)$ is complete. It is sufficient to show that if $\{L_i\}_{i=1}^{\infty}$ is a sequence in $C_Z(X, Y)$ such that $\| L_i \|_Z \leqq (\frac{1}{2})^{2i}$, then $\Sigma_{i=1}^{\infty} L_i$ is $\| \cdot \|_Z$-convergent. Now $\| L \|_Z \geqq \| L \|$ for all $L \in C_Z(X, Y)$, so there is $L \in C(X, Y)$ such that $\lim_{n \to \infty} \| L - \Sigma_{i=1}^{n} L_i \| = 0$. For each i, let (A_i, B_i) be a compact factorization of L_i through Z_i such that $\| A_i \| \| B_i \| \leqq \| L_i \|_Z + (\frac{1}{2})^{2i}$ and $\| A_i \| = \| B_i \|$. Note that $\Sigma_{i=1}^{\infty} \| A_i \| = \Sigma_{i=1}^{\infty} \| B_i \| < \infty$, so $A = \Sigma_{i=1}^{\infty} A_i$ and $B = \Sigma_{i=1}^{\infty} B_iP_i$ exist. Clearly $BA = L$, so L is in $C_Z(X, Y)$. Furthermore,

$$\| L - \sum_{i=1}^{n} L_i \|_Z = \left\| \left(\sum_{i=n+1}^{\infty} B_iP_i \right) \left(\sum_{i=n+1}^{\infty} A_i \right) \right\|_Z \leqq \left(\sum_{i=n+1}^{\infty} \| B_i \| \right) \left(\sum_{i=n+1}^{\infty} \| A_i \| \right) \to 0 .$$

This shows that $\Sigma_{i=1}^{\infty} L_i$ $\| \cdot \|_Z$-converges to L. Q.E.D.

For Banach spaces X, Y and Z and $1 \leqq p < \infty$ let $\pi^p{}_Z(X, Y)$ be the collection of all operators L from X to Y which admit a compact factorization (A, B) through Z with A p-absolutely summing and let $\pi^p{}_Z(L) = \inf\{\pi^p(A) \| B \| : (A, B)$ is a compact factorization of L through $Z\}$. Here $\pi^p(A)$ is the p-absolutely summing norm of A; i.e., $\pi^p(A) = \sup\{[\Sigma_{i=1}^{n} \| A(x_i) \|^p]^{1/p} : \{x_i\}_{i=1}^{n} \subset X$ and for each $x^* \in X^*$, $[\Sigma_{i=1}^{n} | x^*(x_i) |^p] \leqq \| x^* \|^p .\}$.

PROPOSITION 2. *Let X, Y, and Z be Banach spaces, let $1 \leqq p < \infty$, and suppose that $Z \cong \Sigma_p Z$. Then $\pi_Z^p(X, Y)$ is a Banach space under the norm π_Z^p.*

We omit the proof of Proposition 2 because it is very similar to the proof of Proposition 1. The only difficulty in the proof is to show that π_Z^p satisfies the triangle inequality. This step follows from the following simple fact: Let each of $L: X \to Y$ and $M: X \to Z$ be p-absolutely summing. Define $N: X \to (Y \oplus Z)_p$ by $N(x) = (L(x), M(x))$. Then $\pi^p(N) \leqq ([\pi^p(L)]^p + [\pi^p(M)]^p)^{1/p}$.

REMARK. If $Z \cong \Sigma_p Z$, then $\| \cdot \|_Z$ and π_Z^p are uniform cross norms (see [12]) on $X^* \otimes Y$ and thus both $C_Z(X, Y)$ and $\pi_Z^p(X, Y)$ are completed tensor products of X^* and Y.

3. Applications

Let $\{G_i\}_{i=1}^{\infty}$ be a sequence of finite dimensional Banach spaces such that

i) if E is a finite dimensional Banach space and $\varepsilon > 0$ then there exists i such that $d(E, G_i) < 1 + \varepsilon$; and

ii) for each positive integer i there exists an infinite subset J of the positive integers such that $G_i \cong G_j$ for each $j \in J$.

For $1 \leqq p \leqq \infty$, let $C_p = \Sigma_p G_i$. Because of (ii), $C_p \cong \Sigma_p C_p$. We note that C_p is essentially unique in the sense that if $\{G_i'\}_{i=1}^{\infty}$ is another sequence of finite dimensional spaces which satisfies (i) and (ii), then $d(C_p, \Sigma_p G_i') = 1$. It is easy to show that each infinite dimensional subspace of C_p contains a subspace isomorphic to l_p, so C_p and C_r are totally incomparable if $p \neq r$.

PROOF OF THEOREM 1. Suppose $T \in X^* \otimes Y$. Choose i so that $d(\mathscr{R}(T), G_i) < 2$ and let L be an isomorphism of $\mathscr{R}(T)$ onto G_i such that $\| L \| = 1$, $\| L^{-1} \| \leqq 2$. Define $A: X \to C_p$ by $A = LT$ and define $B: C_p \to Y$ by $B = L^{-1} P_i$, where P_i is the natural norm 1 projection of C_p onto G_i. Clearly $BA = T$ and $\| A \| \, \| B \| \leqq 2 \| T \|$. Thus $\| \cdot \|_{C_p}$ is equivalent to the operator norm on $X^* \otimes Y$ and since $X^* \otimes Y$ is dense in $C(X, Y)$, it follows from Proposition 1 that $C_{C_p}(X, Y) = C(X, Y)$. Q.E.D.

PROOF OF THEOREM 2. Suppose first that X is an $\mathscr{L}_p$ space and choose $\lambda \geqq 1$ so that X is an $\mathscr{L}_{p,\lambda}$ space. Let $T \in Y^* \otimes X$ and choose a finite dimensional subspace E of X and a positive integer n so that $d(E, l_p^n) \leqq \lambda$ and $\mathscr{R}(T) \subset E$. Let L be an isomorphism of E onto the span of the first n vectors, $\{\delta_i\}_{i=1}^n$, of the usual basis for l_p such that $\| L \| = 1$, $\| L^{-1} \| \leqq \lambda$ and let P be the natural norm 1 projection of l_p onto $\mathrm{sp}\{\delta_i\}_{i=1}^n$. Then $(LT, L^{-1}P)$ is a compact factorization of T through l_p and $\| T \|_{l_p} \leqq \| LT \| \, \| L^{-1}P \| \leqq \lambda \| T \|$. Thus $\| \cdot \|_{l_p}$ is equivalent to the operator norm on $Y^* \otimes X$ and as in Theorem 1, $C_{l_p}(Y, X) = C(Y, X)$.

Now if $1 < p < \infty$ and X is an $\mathscr{L}_2$ space then X is isomorphic to a complemented subspace of an $\mathscr{L}_p$ space (see [3]) so that the desired conclusion follows from the first part of the proof.

To go the other way, let $1 \leqq r \leqq \infty$ and suppose that $C(C_r, X) = C_{l_p}(C_r, X)$. By the open mapping theorem it follows that there exists a number k such that $\| T \|_{l_p} \leqq k \| T \|$ for all $T \in C(C_r, X)$. Let E be a finite dimensional subspace of X, choose i so that $d(E, G_i) < 2$, and let L be an isomorphism of G_i onto E so that $\| L \| = 1$, $\| L^{-1} \| \leqq 2$. Let P_i be the natural norm 1 projection of C_r onto G_i and let (A, B) be a compact factorization of LP_i through l_p so that $\| A \| \, \| B \| \leqq k + 1$. Note that $\| AL^{-1} \| \, \| B \| \leqq 2(k+1)$ and BAL^{-1} is the identity on E. The desired conclusion now follows from Theorem 4.3 of [4]. Q.E.D.

REMARK. Let $1 \leqq p < \infty$ and let X be an $\mathscr{L}_p$ space, or let $p = \infty$ and let X be a complemented subspace of $C(S)$ for some compact Hausdorff space S. Assume X is infinite dimensional and note that X has the approximation property by [4, Theorem 3]. Now if $1 \leqq p < \infty$, X contains a complemented subspace isomorphic to l_p [3, Proposition 7.3], so that by Theorem 2 every compact operator into X factors compactly through X. If $p = \infty$, then by [6, Theorem 5] X contains a subspace Z isomorphic to l_∞ and hence if T is a compact operator into X, T admits a compact factorization (A, B) through Z. Now B can be extended to a compact operator $\tilde{B}$ from X into X by [4, Theorem 4.1], hence T factors compactly through X. This remark generalizes some results stated by Milman in [5].

PROOF OF THEOREM 3. The hypothesis guarantees that $X^* \otimes Y$ is π^p-dense in the p-absolutely summing operators from X to Y (see [8]). The proof now proceeds along the same lines as the proofs of Theorems 1 and 2. Q.E.D.

PROOF OF THEOREM 4. Let $Y = (X \oplus C_p)_\infty$ and let F be a finite dimensional subspace of Y. A density argument (see, e.g., [2, Lemma 3]) shows that we can assume that $F \subset E \oplus \mathrm{sp} \bigcup_{i=1}^n G_i$ with E a finite dimensional subspace of X. Let T be an operator of finite rank on X such that $T_{|E}$ is the identity on E and $\| T \| \leqq \lambda + 1$. Let $G = \mathscr{R}((I - T)T)$, choose $m > n$ so that $d(G, G_m) < 2$ and let L be an isomorphism of G onto G_m so that $\| L \| = 1$, $\| L^{-1} \| < 2$. Let P be the natural norm 1 projection of C_p onto sp $\bigcup_{i=1}^{n} G_i$ and let P_m be the natural norm 1 projection of C_p onto G_m. Define $S: Y \to Y$ by

$$S(x,y) = (T(x) + L^{-1}P_m(y),\ L(I-T)T(x) + L(I-T)L^{-1}P_m(y) + P(y)).$$

(i) If $x \in E$, then $T(x) = x$ and $(I-T)T(x) = 0$, so $S(x,0) = (x,0)$. If $y \in \operatorname{sp} \bigcup_{i=1}^{n} G_i$, then $P(y) = y$ and $P_m(y) = 0$, so $S(0,y) = (0,y)$. Thus $S_{|E \oplus \operatorname{sp} \cup_{i=1}^{n} G_i}$ is the identity.

(ii) If $x \in X$,
$$\begin{aligned} SS(x,0) &= S(T(x), L(I-T)T(x)) \\ &= (TT(x) + (I-T)T(x), L(I-T)T(T(x)) + L(I-T)(I-T)T(x) + 0) \\ &= (T(x), L(I-T)T(x)) = S(x,0). \end{aligned}$$
If $y \in C_p$,
$$\begin{aligned} SS(0,y) &= S(L^{-1}P_m(y), L(I-T)L^{-1}P_m(y) + P(y)) \\ &= (TL^{-1}P_m(y) + (I-T)L^{-1}P_m(y) + 0, L(I-T)TL^{-1}P_m(y) \\ &\quad + L(I-T)(I-T)L^{-1}P_m(y) + 0 + 0 + PP(y)) \\ &= (L^{-1}P_m(y), L(I-T)L^{-1}P_m(y) + P(y)) = S(0,y). \end{aligned}$$
Thus S is a projection.

(iii) $\| S \| \leqq \max\{(\lambda+1)+2,\ (\lambda+2)(\lambda+1)+(\lambda+2)2+1\} \equiv \lambda'$. Combining (i), (ii), and (iii) with the obvious fact that S has finite rank, we conclude that Y is a $\pi_{\lambda'}$ space. Q.E.D.

Problem. Suppose that X has the $1+\varepsilon$-metric approximation property for every $\varepsilon > 0$. Then is $(X \oplus C_p)_\infty$ a $\pi_{1+\varepsilon}$ space for every $\varepsilon > 0$?

The reason that this problem is interesting is that it is known (see [1]) that a separable space which is a $\pi_{1+\varepsilon}$ space for every $\varepsilon > 0$ must admit a Schauder decomposition into finite dimensional subspaces, while it is unknown for general λ whether a separable π_λ space must admit such a decomposition.

Added in Proof. Theorem 4 and the results of [13] show that if X is separable and X^* has the λ-metric approximation property for some $\lambda \geqq 1$, then $X \oplus C_p$ has a Schauder basis; moreover, if also X^* is separable and $1 < p \leqq \infty$, then $X \oplus C_p$ has a shrinking Schauder basis. To see this, note first that $(X \oplus C_p)^*$ is a $\pi_{\lambda'}$ space for some $\lambda' \geqq 1$. Indeed, if $1 < p \leqq \infty$, this follows from Theorem 4 and the canonical isomorphism $(X \oplus C_p)^* = X^* \oplus C_q$, where $1/p + 1/q = 1$. (If $p = 1$, a simple modification of Theorem 4 is necessary). It then follows from

Theorem 1.3 of [13] that $X \oplus C_p$ has a finite dimensional Schauder decomposition, which can be chosen to be shrinking if $(X \oplus C_p)^*$ is separable; i.e., if X^* is separable and $1 < p \leqq \infty$. It then follows from the proof of Corollary 4.12 of [13] that there is a sequence $\{X_i\}_{i=1}^{\infty}$ of finite dimensional Banach spaces such that $(X \oplus C_p) \oplus \Sigma_p X_i$ has a Schauder basis, which can be taken to be shrinking if X^* is separable and $1 < p \leqq \infty$. But clearly $C_p \oplus \Sigma_p X_i$ is isomorphic to C_p, hence $(X \oplus C_p) \oplus \Sigma_p X_i$ is isomorphic to $X \oplus C_p$. In particular:

A. *The reflexive spaces C_p $(1 < p < \infty)$ have Schauder bases and if X is any separable reflexive space which has the λ-metric approximation property for some $\lambda \geqq 1$, then the reflexive space $X \oplus C_p$ has a Schauder basis.*

B. *The subspace C_∞ of l_∞ has a shrinking Schauder basis, and if X is any Banach space such that X^* is separable and has the λ-metric approximation property for some $\lambda \geqq 1$, then $X \oplus C_\infty$ has a shrinking Schauder basis.*

REMARKS. 1) It is noted in [13] that if X^* is separable and has the approximation property, then X^* has the 1-metric approximation property.

2) It seems likely that if X is separable and has the λ-metric approximation property for some $\lambda \geqq 1$, then $X \oplus C_p$ has a Schauder nasis, but I cannot prove this.

3) [14] and [15] contain results related to Theorem 4, A, and B above.

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